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Stabilization of a Chain of Three Integrators Subject to a Phase Constraint

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Abstract—The problem of stabilizing a chain of three integrators subject to a phase constraint is studied. Continuous constrained control in the form of nested sigmoids, which guarantees the fulfillment of the phase constraint, is synthesized. A Lyapunov function is constructed, and necessary and sufficient conditions of global stability of the closed-loop system are established. The discussion is illustrated by numerical examples.

Keywords: stabilization of a chain of three integrators, global stability, phase constraint, nested sigmoids, Lyapunov function

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1. INTRODUCTION

The problem of stabilizing a chain of three integrators subject to a phase constraint by means of a continuous control is studied. Stabilization of chains of integrators is one of the topical control problems, which has been widely discussed in the literature during last several decades (see, e.g., [1, 2] and references therein). The interest to this problem is due to the fact that original models in many applications are specified as chains of integrators and the controls developed for chains of integrators are easily extended to other classes of systems.

Among the variety of stabilizing controls applied to solving this problem, the class of feedbacks in the form of nested (both smooth and non-smooth) saturation functions can be distinguished [2–14]. The interest to such feedbacks is explained by the number of remarkable properties of the closed-loop system obtained: they automatically take into account boundedness of the control resource and ensure fulfillment of certain phase constraints, which is especially important far from the equilibrium state, as well as guarantee exponential rate of the deviation decrease near the equilibrium [3–7]. Note also the use of such feedbacks in the problems related to the adjustment of coefficients in the robust control laws [8].

The use of feedbacks in the form of nested saturation functions gives rise to study of quite complicated nonlinear systems (in the case of non-smooth saturation functions, these are linear switching systems), stability analysis of which is a nontrivial task. Global stability has been proved mainly for second-order systems with nested saturators [3, 5, 9] and sigmoids [3, 10]. Practically in all works studying systems of order three or higher, only local stability was proved [3, 4, 11, 12]. In rare cases of feedbacks of special form, global stability has been established for systems of order three [12] (piecewise continuous control) or four [13] (impulse control). As far as the authors know, the problem of global stability for the general case of n nested saturators was considered only in the works by A. Teel [2, 14]. However, global stability has been proved only in the case where limit values of the nested saturators satisfy certain inequalities, which are seldom fulfilled in

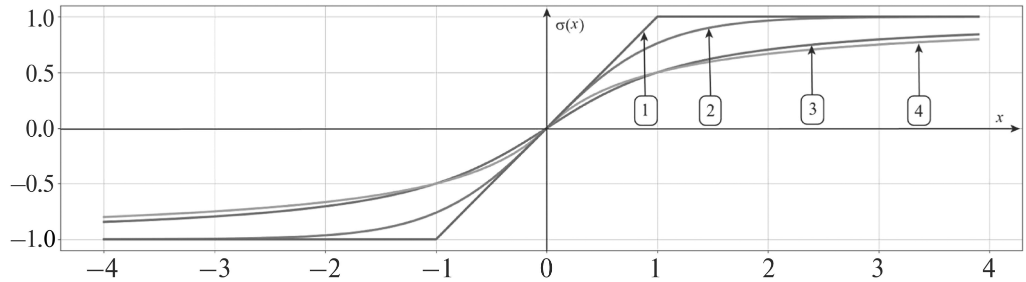

Fig. 1. Examples of saturation functions: $\text{sat}(x)$ (1); $\tanh(x)$ (2); $2\arctan(x)/\pi$ (3); $x/(1+|x|)$ (4).

practice [2, Theorem 2.1]. The authors are not aware of works (except for abovementioned Teel's papers) where global stability were proved for a system of order three or higher stabilized by a continuous control guaranteeing fulfillment of a phase constraint.

Saturation function is a continuous nondecreasing function $S(x)$ of scalar variable that has finite limits when $x \to \pm\infty$. Among the saturation functions, the class of smooth strongly increasing functions called *sigmoids* can be distinguished [15]. In the literature, one can meet several slightly differing definitions of the sigmoids. We will use the following

Definition 1. Sigmoid is a smooth strongly increasing odd function of scalar variable $\sigma(x)$ satisfying the following conditions:

(a) $\sigma(x) \to \pm 1$ as $x \to \pm\infty$;
(b) $\max_x \sigma'(x) = \sigma'(0)$;
(c) $\sigma'(0) = 1$.

Functions satisfying the above definition but having different from ones limits at infinity and derivative at zero are referred to as *sigmoid functions.* Any sigmoid function $S(x)$ can be constructed from a sigmoid $\sigma(x)$ by specifying two coefficients: $S(x) = k_2\sigma(k_1 x)$, $k_1, k_2 > 0$. It is easy to see that, for any two sigmoid functions $S_1(x)$ and $S_2(x)$, $S(x) = S_1(S_2(x))$ is also a sigmoid function. When proving global stability, we will need the inequalities

$$S(x)x > 0 \ \forall x \neq 0, \tag{1}$$

$$[S(x + x_0) - S(x_0)]x > 0 \ \forall x \neq 0, \ \forall x_0, \tag{2}$$

which directly follow from the definition of the sigmoid.

The family of the sigmoid functions includes error function, arctangent, hyperbolic tangent, and other functions of similar form. The limit case of the sigmoid is the non-smooth saturation function called *saturators*: $\text{sat}(x) = x$ when $|x| \leqslant 1$ and $\text{sat}(x) = \text{sgn}(x)$ when $|x| > 1$. Examples of the saturation functions are shown in Fig. 1. Other examples of the saturation functions and discussions of their properties can be found in [15]. In the control problems, the hyperbolic tangent is most often used as a sigmoid since it approximates the saturator better than other smooth saturation functions and, moreover, its derivatives are expressed in terms of the function itself. In the framework of this study, it does not matter what sigmoid is used in the feedback, since the proof of global stability is valid for any functions satisfying the above definition.

In this work, we suggest to stabilize a chain of three integrators by means of a special feedback including two nested sigmoids. The goal of the study is to prove global stability of the closed-loop system obtained under certain simple conditions on the feedback coefficients.

2. PROBLEM STATEMENT

We consider the problem of stabilizing a third-order integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = U(x), \quad x \equiv [x_1, x_2, x_3]^{\mathrm{T}}, \tag{3}$$

at the origin by means of a smooth feedback $U(x)$ guaranteeing the fulfillment of the phase constraint

$$|x_3(t)| \leqslant X_3. \tag{4}$$

Such a statement naturally comes to existence in many applications, for example, when stabilizing a mechanical system [11], where state variables are position, velocity, and traction (acceleration) and the system is controlled by varying the traction (e.g., by means of a step motor). A similar system with the phase constraint on the third variable, but with a discontinuous control, was considered in [16]. Since the traction in real systems is limited, the stabilizing control must not result in the violation of the phase constraint (4), where X_3 is the maximum possible traction.

The stabilizing control is sought in the form

$$U(x) = -k_5(x_3 + k_4\sigma_2(k_3(x_2 + k_2\sigma_1(k_1x_1)))), \tag{5}$$

where σ_1 and σ_2 are arbitrary sigmoids. The feedback of this form guarantees the fulfillment of phase constraint (4) with $X_3 = k_4$ if $|x_3(0)| \leqslant k_4$. Indeed, suppose that the phase constraint is satisfied at the initial moment. Variable $x_3(t)$ achieves local extremum on the trajectory when $U(x) = 0$; on the other hand, from formula (5), it is seen that the control equals zero when $x_3 = -k_4\sigma_2(\cdot)$. Hence, $|x_3(t)|$ cannot be greater than k_4; i.e., domain $|x_3| \leqslant k_4$ is an invariant set of the system. Thus, if variable x_3 cannot physically exceed its limit value (like, for instance, in the abovementioned example of the mechanical system), then it is sufficient to study stability of the system in this invariant set. We, however, consider a more general problem statement and will prove stability for any initial conditions in R^3. In so doing, if the initial point belongs to the invariant set, then the phase constraint (4) is fulfilled for any $t \geqslant 0$; otherwise, starting from some (depending on the initial conditions) finite instant.

Additional advantages of control (5) are (a) exponential rate of the deviation decrease near the equilibrium and (b) its boundedness for any deviations from the equilibrium state as long as the phase constraint is fulfilled at the initial point.

Coefficients k_2 and k_4, which set limits of sigmoid variations, are referred to as *model parameters,* since their values are determined by the model of the system under study, and, unlike the other three coefficients cannot be selected arbitrarily. Given k_2 and k_4, parameters k_1, k_3, and k_5 determine the character of the transition process [5, 7] and are referred to as *design parameters.* They are selected by the designer of the control system with the aim, for instance, to optimize (in one or another sense) its performance.

Without loss of generality, the model parameters can be set equal to ones, which reduces the number of system parameters to three. Indeed, let us turn to the dimensionless model by applying the same change of variables and time as in the two-dimensional case [5], i.e., $\tilde{t} = k_4t/k_2$, $\tilde{x}_1 = k_4x_1/k_2^2$, and $\tilde{x}_2 = x_2/k_2$, and define the third dimensionless variable as $\tilde{x}_3 = x_3/k_4$. Substituting the new variables into system (3), (5) and turning to differentiation with respect to the dimensionless time, we obtain the dimensionless model in which $\tilde{k}_2 = \tilde{k}_4 = 1$ and three other coefficients are given by the formulas $\tilde{k}_1 = k_1k_2^2/k_4$, $\tilde{k}_3 = k_2k_3/k_4$, and $\tilde{k}_5 = k_2k_5/k_4$. In what follows, we assume that all variables and parameters are dimensionless and will use the same notation (without tilde) for them. In the dimensionless model, feedback (5) takes the form

$$U(x) = -k_5(x_3 + \sigma_2(k_3(x_2 + \sigma_1(k_1x_1)))). \tag{6}$$

The goal of the study is to determine the conditions on the coefficients for which the proposed feedback stabilizes the system in the entire space, i.e., to establish conditions of global stability of system (3), (6). The study of stability presented in the next section is based on the construction of an integral Lyapunov function of the closed-loop system. We will prove that the necessary conditions of stability of the linearized in the neighborhood of the origin system are sufficient for its global stability. Note that the application of other known approaches to studying stability, for example, those based on the construction of the Lurie–Postnikov function or on the immersion into the class of linear nonstationary systems with subsequent application of methods of absolute stability theory allows one to prove, as a rule, only local stability (even if the system under study is stable in the whole) and construct an estimate of the invariant attraction domain.

3. GLOBAL STABILITY CONDITIONS

Theorem 1. *System* (3), (6), *where* $\sigma_1(\cdot)$ *and* $\sigma_2(\cdot)$ *are arbitrary sigmoids, is globally asymptotically stable if and only if all the feedback coefficients are positive and* $k_5 > k_1$.

Proof. *Necessity.* In order that the system be globally stable, it is necessary that the linearized in the neighborhood of the origin system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -k_5x_3 - k_5k_3x_2 - k_5k_3k_1x_1$$

be stable. Applying the Hurwitz criterion to the latter system, we find that it is stable when all the coefficients are positive and the condition $k_5 > k_1$ holds. *Sufficiency.* Let coefficients k_1, k_3 and k_5 be positive. Let us consider the function

$$V(x) = k_5^2\int\limits_0^{x_1}\sigma_2(k_3\sigma_1(k_1s))ds + k_5\int\limits_0^{x_2}\sigma_2(k_3(s+\sigma_1(k_1x_1)))ds + \frac{1}{2}(x_3+k_5x_2)^2 \tag{7}$$

and prove that it is Lyapunov function of system (3), (6).

Let Φ_1 and Φ_2 denote the first and second, respectively, integral terms in (7). Let us prove that their sum and, hence, the entire function $V(x)$ are positive $\forall x \in R^3$.

Let us transform the second term Φ_2 by changing the integration variable $\tilde{s} = s + \sigma_1(k_1x_1)$:

$$\Phi_2 = k_5\int\limits_{\sigma_1(k_1x_1)}^{x_2+\sigma_1(k_1x_1)}\sigma_2(k_3\tilde{s})d\tilde{s} = k_5\int\limits_0^{x_2+\sigma_1(k_1x_1)}\sigma_2(k_3\tilde{s})d\tilde{s} - k_5\int\limits_0^{\sigma_1(k_1x_1)}\sigma_2(k_3\tilde{s})d\tilde{s}.$$

In the second term on the right-hand side of the last formula, we perform implicit one-to-one (by virtue of monotonicity of function σ_1) change of the integration variable $\tilde{s} = \sigma_1(k_1s)$. Taking into account that $d\tilde{s} = k_1\sigma_1'(k_1s)ds$, where the prime denotes differentiation with respect to the argument, the sum of Φ_1 and Φ_2 takes the following form:

$$\Phi_1 + \Phi_2 = k_5\int\limits_0^{x_1}\sigma_2(k_3\sigma_1(k_1s))[k_5 - k_1\sigma_1'(k_1s)]ds + k_5\int\limits_0^{x_2+\sigma_1(k_1x_1)}\sigma_2(k_3\tilde{s})d\tilde{s}.$$

The second integral on the right-hand side of this formula is positive by virtue of (1). Since the derivative of the sigmoid satisfies the condition $\sigma'(s) \leqslant 1$ and, by the assumption of the theorem, $k_5 > k_1$, we have

$$k_5 - k_1\sigma_1'(k_1s) > 0, \tag{8}$$

from which it follows that the first integral and, hence, function $V(x)$ are positive for all $x \neq 0$.

It is evident that $V(x)$ tends to infinity as $||x|| \to \infty$. Further, differentiating $V(x)$ by virtue of system (3), (6) and omitting the argument k_1x_1 of functions σ_1 and σ_1' to shorten the notation, we obtain

$$\dot V = k_5^2\sigma_2(k_3\sigma_1(\cdot))x_2 + k_5x_2\int\limits_0^{x_2}\sigma_2'(k_3(s+\sigma_1(\cdot)))k_3\sigma_1'(\cdot)k_1ds$$

$$+\,k_5\sigma_2(k_3(x_2+\sigma_1(\cdot)))x_3 + (x_3+k_5x_2)[-k_5(x_3+\sigma_2(k_3(x_2+\sigma_1(\cdot))) + k_5x_3]$$

$$= k_5^2\sigma_2(k_3\sigma_1(\cdot))x_2 - k_5^2\sigma_2(k_3(x_2+\sigma_1(\cdot)))x_2 + k_1k_3k_5\sigma_1'(\cdot)x_2\int\limits_0^{x_2}\sigma_2'(k_3(s+\sigma_1(\cdot)))ds.$$

Let us transform the integral on the right-hand side of the last expression:

$$\int\limits_0^{x_2}\sigma_2'(k_3(s+\sigma_1(\cdot)))ds = \int\limits_{\sigma_1(\cdot)}^{x_2+\sigma_1(\cdot)}\sigma_2'(k_3\tilde s)d\tilde s = \frac{1}{k_3}\sigma_2(k_3(x_2+\sigma_1(\cdot))) - \frac{1}{k_3}\sigma_2(k_3\sigma_1(\cdot)).$$

Substituting the expression obtained into the formula for $\dot V(x)$, we get

$$\dot V(x) = k_5\sigma_2(k_3\sigma_1(\cdot))x_2(k_5 - k_1\sigma_1'(\cdot)) - k_5\sigma_2(k_3(x_2+\sigma_1(\cdot)))x_2(k_5 - k_1\sigma_1'(\cdot))$$
$$= -k_5(k_5 - k_1\sigma_1'(\cdot))[\sigma_2(k_3(x_2+\sigma_1(\cdot))) - \sigma_2(k_3\sigma_1(\cdot))]x_2.$$

The product of the expression in the square brackets and x_2 is positive by virtue of (2), from which, with regard to (8), it follows that the derivative is negative definite for any $x_2 \neq 0$. The derivative vanishes only on the set $x_2 = 0$, which contains no entire trajectories but $x = 0$.

Thus, function $V(x)$ satisfies all the conditions of the Barbashin–Krasovski theorem [20], and, hence, the origin is asymptotically stable equilibrium of system (3), (6) in the whole. The theorem is proved.

4. NUMERICAL EXAMPLES

As an illustration, we present results of numerical calculations for the feedback (6) in the form of nested hyperbolic tangents with the coefficients $k_1 = 1$, $k_3 = 3$, and $k_5 = 5$. Figure 2 shows the invariant set of the system bounded by the level surface of the Lyapunov function (7) $V(x) = k_5^2$.

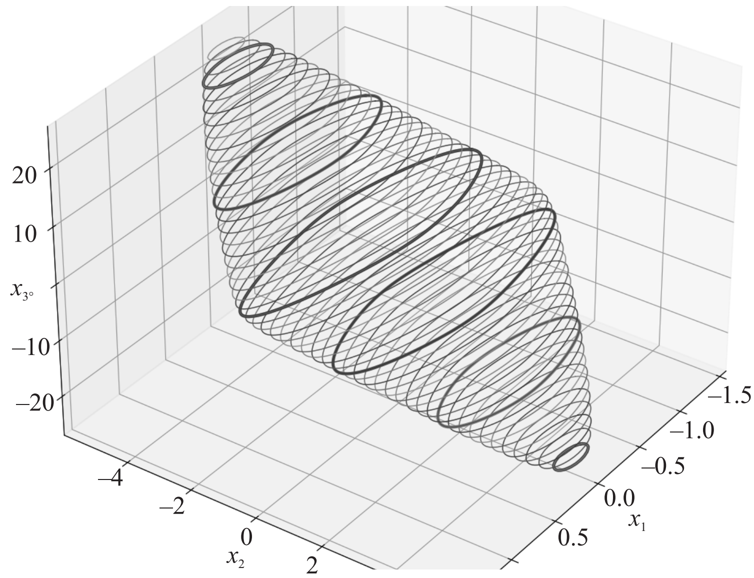

Fig. 2. Level surface of the Lyapunov function (7).

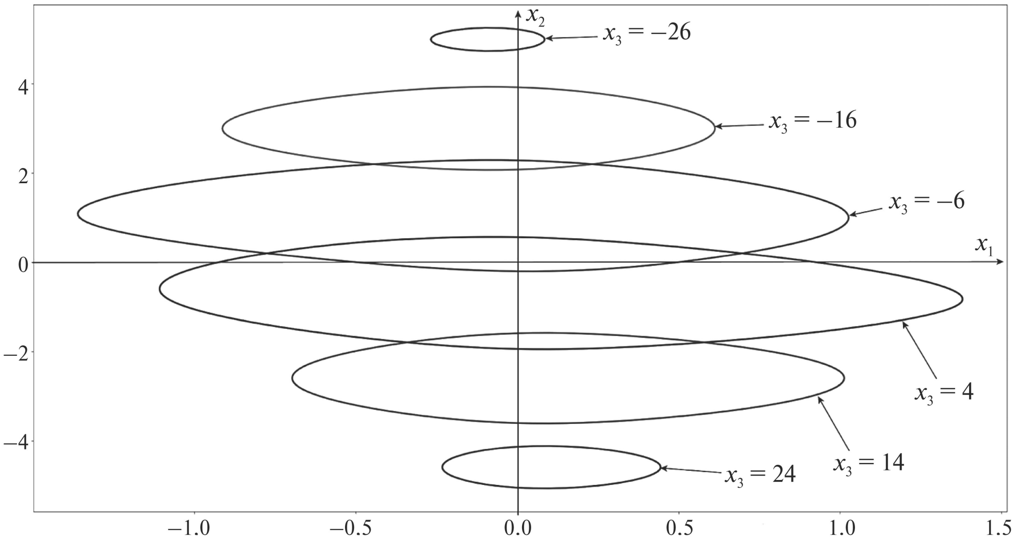

Fig. 3. Projections of cross-sections of the invariant set by planes $x_3 = \text{const}$ onto the plane (x_1, x_2).

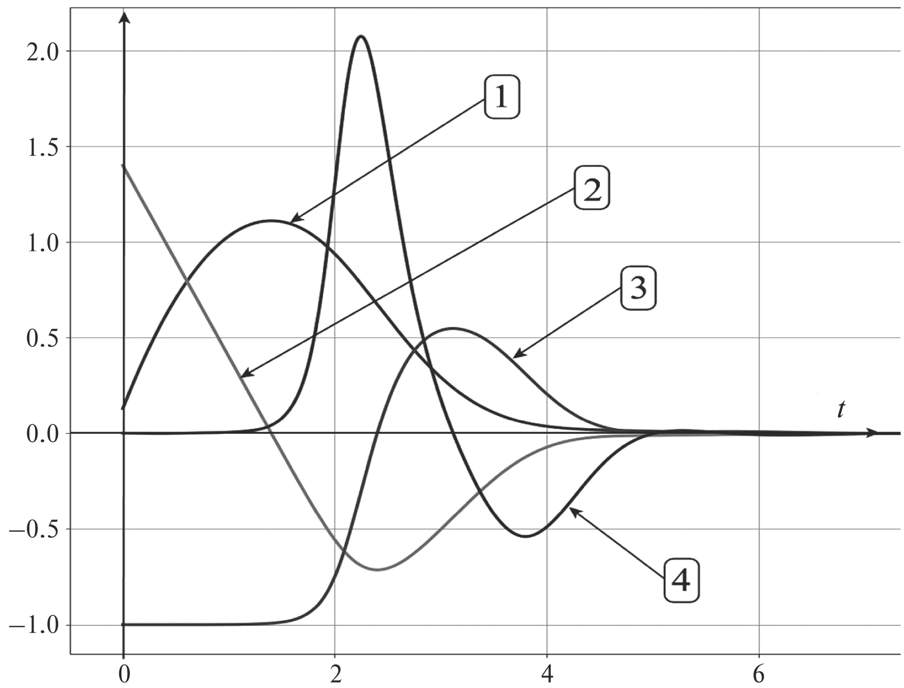

Fig. 4. Plots of deviation $x_1(t)$ (1), velocity $x_2(t)$ (2), acceleration $x_3(t)$ (3), and control $U(t)$ (4).

For greater clarity, Fig. 3 shows projections of six cross-sections of the level surface onto the plane (x_1, x_2) (in Fig. 2, these cross-sections are depicted by bold lines) by the planes $x_3 = c_i$, $c_1 = -26$, $c_2 = -16$, $c_3 = -6$, $c_4 = 4$, $c_5 = 14$, and $c_6 = 24$.

Results of solving stabilization problem for the system with initial conditions $x_1(0) = 0.1$, $x_2(0) = 1.4$, $x_3(0) = -1$ are presented in Fig. 4, which demonstrates efficiency of the stabilization. The curves marked by 1, 2, 3, and 4 are plots of dependencies of deviation x_1, velocity x_2, acceleration x_3, and control U, respectively, on time. Although at the initial instant, the system moves in the direction opposite to the equilibrium state, the deviation, after natural growth at the initial stage, rapidly (exponentially) decreases, the phase constraint is fulfilled for any $t \geqslant 0$, control is reasonably constrained and does not result in overshooting.

5. CONCLUSIONS

The problem of stabilizing a chain of three integrators by a continuous control that guarantees the fulfillment of a phase constraint on the third state variable has been studied. By turning to dimensionless state variables, the original problem depending on five feedback coefficients has been reduced to study of a three-parameter system. Advantages of the proposed feedback in the form of nested sigmoids have been discussed. The basic result of the work is construction of the Lyapunov function by means of which sufficient conditions of global stability of the closed-loop system have been established. Numerical examples illustrating efficiency of stabilization by means of the proposed feedback have been presented.

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